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Analysis of Hamiltonian Dynamics of Dispersion-Managed Coupled Breathers in Optical Transmission System

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Abstract

We theoretically derive the Hamiltonian dynamics of dispersion-managed coupled breathers in optical transmission system as a result of phases-plane dynamics of the system. We analytically show that the contribution of a perturbation term caused by a noise sourced by amplifiers in optical fiber can influence the amplitudes and chirps of the pulses dynamics. We also find that the coupled breathers dynamics depend on a certain relationship of the chirps and amplitudes. The results of the perturbed coupled **NLS** are then reduced to that in the unperturbed system and the single **NLS** case.

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I. Introduction

A dispersion-managed (**DM**) optical fiber is designed to create a low (or even zero) path-averaged dispersion by periodically alternating dispersion sign along an optical line which dramatically reduces pulse broadening. Recently dispersion management has become an essential technology for development of ultrafast high-bit rate optical communication lines. There are some effective methods to improve the performance of a soliton transmission system such as the use of a dispersion-compensation technique, the use of dispersion-managed technique and so on.

Analytic techniques about the dispersion managed system of the single **NLS** by Kutz et. al.¹ and Gabitov et al.² have been separately developed to describe the amplitude and phase fluctuations that occur on the scale of the dispersion map. The aim of the present paper is to extend such analytic descriptions of leading-order amplitude and phase fluctuations in the dispersion-managed coupled breathers derived from the Gaussian pulses as a solution of the coupled **NLS** with a perturbation term of a noise sourced by amplifier. We use an extended average variational method to reduce the complicated coupled **NLS** with a periodic dispersion map to several set of complicated nonlinear ordinary differential equations (**ODEs**), which accurately predict the completed coupled breathers dynamics. In order to find the Hamiltonian dynamics of this system, the **ODEs** are derived by reducing and finding the parameters amplitudes $\eta_j(Z)$, and chirps $\beta_j(Z)$ in which frequencies $f_j(Z)$ and center position of the pulses $\mathbb{T}_{0j}(Z)$ have been set to be identical with zero values.

The paper is organized as follows. In **Section 2** we introduce the appropriate extended **coupled NLS with a perturbation term** and its relevant physical parameters. In **Section 3** we derive the Lagrangian of the nonintegrable coupled **NLS** with the perturbation terms using variational method, and find that in order to make the method work in this system, the parameters frequencies $f_j(Z)$ and center position of the pulses $\mathbb{T}_{0j}(Z)$ must be chosen to be very small (≈ 0). **Section 4** transforms the extended coupled **NLS** equations from Lagrangian to Hamiltonian formulation for which the solution of the equations of the coupled breathers dynamics can be expressed in terms of a modified quadrature. **Section 5** contains of phase-plane dynamics of the derived coupled breathers dynamical system. **Section 6** is the reduced results of section 5 related to the single **NLS** provided by Kutz, et al., and the unperturbed coupled **NLS**. These system solutions are found by reducing some parameters of the coupled **NLS** which has the perturbation term. A conclusion of the results is provided in **Section 7**.

II. Formulation of the Nonintegrable Coupled NLS with a Perturbation Term

A more complicated propagation of optical pulse in **DM** fiber is described by a non-integrable coupled nonlinear Schrödinger equation (**NLS**) with peri-

odically varying dispersion $\sigma(Z)$ and some additional perturbation terms:

$$\begin{aligned} i\frac{\partial Q_1}{\partial Z} + \frac{\sigma_1(Z)}{2}\frac{\partial^2 Q_1}{\partial T^2} + \left(|Q_1|^2 + |Q_2|^2\right)Q_1 &= iG(Z)Q_1 + i\epsilon_1 R_1 \\ i\frac{\partial Q_2}{\partial Z} + \frac{\sigma_2(Z)}{2}\frac{\partial^2 Q_2}{\partial T^2} + \left(|Q_1|^2 + |Q_2|^2\right)Q_2 &= iG(Z)Q_2 + i\epsilon_2 R_2 \end{aligned} \quad (1)$$

where Q_1 and Q_2 are the electric-field envelopes normalized by the peak field powers $|E_{01}|^2$ and $|E_{02}|^2$, $G(Z) = -\Gamma + G_a \sum_{n=1}^M \delta(Z - nZ_a)$ represents the effects of the fiber loss and the amplification, where Γ is the normalized fiber loss, $G_a(Z) (\equiv \Gamma Z_a)$ is the gain of amplifier, the following term³

$$\epsilon_j R_j = r_{a_j} \exp \left\{ i \left[f_j(Z) (T - \mathbb{T}_{0_j}(Z)) + \frac{\phi_j(Z)}{2} \right] \right\}, j = 1, 2$$

are noise factors of the amplifiers ($1 \gg \epsilon$) where r_{a_j} , $f_j(Z)$, $\mathbb{T}_{0_j}(Z)$ and $\phi_j(Z)$ are complex amplitude of the noises, frequencies of the noises, center position of the pulses, and the phases of the pulses, respectively and we have set $f_j(Z)$, and $\mathbb{T}_{0_j}(Z)$ to be very small, and $\sigma_j(Z)$ is a periodic dispersion map. To follow up, the previous work of Kutz, Holmes, *et al.*, in their single **NLS** case, we define the variable T as the physical time normalized by $T_0/1.76$, where T_0 is the full width at half maximum (**FWHM**) of the pulse ($T_0 = 50$ ps). The variable Z represents the physical distance divided by the dispersion length Z_0 , which corresponds to the average value (\overline{D}) of the dispersion map $\sigma(Z)$. Thus we find

$$Z_0 = \frac{2\pi c}{\lambda_0 \overline{D}} \left(\frac{T_0}{1.76} \right) \quad (2)$$

which gives

$$|E_{01}|^2 = |E_{02}|^2 \approx \frac{\lambda_0 A_{eff}}{2\pi n_2 Z_0} \quad (3)$$

for the peak field intensities of the one-soliton solution corresponding to the average dispersion. Here $n_2 = 2.6 \times 10^{-16}$ cm²/W is the nonlinear coefficient of the fiber, $A_{eff} = 55$ μm^2 is the effective cross-sectional area of the fiber, and $\lambda_0 = 1.55$ μm and c are the carrier's free-space wavelength and speed of light, respectively. In the present analysis, we involve the periodic intensity fluctuations that are due to fiber loss and amplification because they have very small perturbations to the average pulse dynamics. Thus the peak powers given in eq.(3) are almost the same as the path-average peak power of the propagating pulses. Note that we also consider long-length scale fluctuations in intensity that are due to gain-shape changes and saturation effects of the small noises in the amplifiers.

The parameters that are of greatest interest correspond to the particular dispersion map that is being used. It is helpful therefore to consider the specific case of a piecewise constant step-function dispersion map such that

$$\sigma_1(Z) = \sigma_2(Z) = \frac{1}{\overline{D}} \begin{cases} D_-, & 0 < Z < \frac{1}{2} \frac{Z_-}{Z_0} \\ D_+, & \frac{1}{2} \frac{Z_-}{Z_0} < Z < \frac{1}{2} \frac{Z_-}{Z_0} + \frac{Z_+}{Z_0} \\ D_-, & \frac{1}{2} \frac{Z_-}{Z_0} + \frac{Z_+}{Z_0} < Z < P = \frac{Z_+ + Z_-}{Z_0} \end{cases} \quad (4)$$

where $\sigma(Z) = \sigma(Z + P)$, $D_+ > 0 > D_-$ are the dispersions in each segment of fiber (in picoseconds per kilometer.nanometer), Z_{\pm} is the length of each fiber segments (in kilometers), and the average dispersion is simply given by

$$\overline{D} = \frac{D_+ Z_+ + D_- Z_-}{Z_- + Z_+}. \quad (5)$$

We also follow the values of the parameters ($\overline{D} = 0.2$ ps/km.nm, $Z_- = 450$ km, $Z_+ = 60$ km, $D_- = -2.1$ ps/km.nm, and $D_+ = 17.45$ ps/km.nm) which are used by Kutz *et al.*, in their explanation related to the single **NLS**.

In order to simplify eq.(1), we use a transformation as follows

$$Q_j(Z, T) = \sqrt{\Upsilon(Z)} U_j(Z, T), \quad j = 1, 2. \quad (6)$$

We then find

$$\begin{aligned} i \frac{\partial U_1}{\partial Z} + \frac{\sigma(Z)}{2} \frac{\partial^2 U_1}{\partial T^2} + \Upsilon(Z) (|U_1|^2 + |U_2|^2) U_1 &= \frac{i \epsilon_1 R_1}{\sqrt{\Upsilon(Z)}} \\ i \frac{\partial U_2}{\partial Z} + \frac{\sigma(Z)}{2} \frac{\partial^2 U_2}{\partial T^2} + \Upsilon(Z) (|U_1|^2 + |U_2|^2) U_2 &= \frac{i \epsilon_2 R_2}{\sqrt{\Upsilon(Z)}} \end{aligned} \quad (7)$$

where the reduced noise factor used in this investigation is

$$\epsilon_j R_j = \sqrt{\eta_j} r_{a_j} \exp \left\{ i \frac{\phi_j(Z)}{2} \right\},$$

in which the amplitude factors $\sqrt{\eta_j}$ used here are to reduce the influence of the noise in the chirp dynamics and $\Upsilon(Z)$ is chosen by following the work of Okamawari, *et al.*³ in the case of their single **NLS** of eq.(7),

$$\Upsilon(Z) \equiv \Upsilon(0) \exp \left[2 \int_0^Z G(Z') dZ' \right], \quad (8)$$

and the average values of $\Upsilon(Z)$ and $\sigma(Z)$ are assumed to be unity in the normalized unit. The imaginary term in the perturbation term can also be made as a real number by choosing, for instance, the complex amplitude of the noises $r_{a_j} = \exp(-0.5i\phi_j(Z))$.

III. Variational Method of the Coupled NLS

The fundamental physical principle involved in this approach is derived from Lagrangian methods or Euler-Lagrangian equation for classical mechanics written by Goldstein, H.⁴ This requires the dynamics to minimize a certain energy

integral closely related to in Hamilton's principle. We can restate the extended coupled **NLS** eq.(7) in terms of its variational form by defining the Lagrangian

$$L = \int_{-\infty}^{\infty} \mathcal{L}(U_j, U_j^*) dT, \quad j = 1, 2 \quad (9)$$

where $\mathcal{L}(U_j, U_j^*)$ is the Lagrangian density for the eq.(7) given by

$$\begin{aligned} \mathcal{L}(U_j, U_j^*) &= \sum_{j=1}^2 \left[i \left(U_j \frac{\partial U_j^*}{\partial Z} - U_j^* \frac{\partial U_j}{\partial Z} \right) + \sigma(Z) \left| \frac{\partial U_j}{\partial T} \right|^2 - \Upsilon(Z) |U_j|^4 \right] \\ &\quad - \sum_{j=1}^2 \frac{2i\epsilon_j R_j}{\sqrt{\Upsilon(Z)}} (U_j - U_j^*) - \left(2\Upsilon(Z) |U_1|^2 |U_2|^2 \right). \end{aligned} \quad (10)$$

In order to reproduce eq.(7) and its complex conjugate, the Lagrangian density must be determined by requiring the following equation

$$\frac{\delta \mathcal{L}(U_j, U_j^*)}{\delta U_j^*} = \frac{\delta \mathcal{L}(U_j, U_j^*)}{\delta U_j} = 0. \quad (11)$$

In the reduced model developed here, we assume that the fundamental dynamics occurs owing to changes in amplitudes and quadratic phase chirps and they depend on very small noises sourced by amplifiers. Further, we assume a solution of eq.(7) that has an extended Gaussian shape as follows

$$U_j(Z, T) = A_j b_j \sqrt{\eta_j} \exp \left\{ -(\kappa_j \eta_j T)^2 + i \left[\beta_j \kappa_j^2 T^2 + \frac{\phi_j(Z)}{2} \right] \right\}, \quad j = 1, 2 \quad (12)$$

where $\eta_j(Z)$ and $\beta_j(Z)$ are free parameters that correspond to the amplitude and quadratic-chirp dynamics, respectively. We then find the total energy pulses as follows

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(|Q_1(Z, T)|^2 + |Q_2(Z, T)|^2 \right) dT \\ &= \int_{-\infty}^{\infty} \left[(b_1^2 \exp(-2\kappa_1^2 T^2)) + (b_2^2 \exp(-2\kappa_2^2 T^2)) \right] dT \\ &= \frac{b_1^2}{\kappa_1} \sqrt{\frac{\pi}{2}} + \frac{b_2^2}{\kappa_2} \sqrt{\frac{\pi}{2}} \end{aligned} \quad (13)$$

As with the governing eq.(1), this ansatz conserves the total energy of the pulse, i.e., $\int_{-\infty}^{\infty} (|Q_1|^2 + |Q_2|^2) dT = \left(\frac{A_1^2 b_1^2}{\kappa_1} \sqrt{\frac{\pi}{2}} \right) + \left(\frac{A_2^2 b_2^2}{\kappa_2} \sqrt{\frac{\pi}{2}} \right) = \text{constant}$. More generally, any localized ansatz for which the amplitude and width are related

as in eq.(12) can be utilized. Related to the case in the same width (**FWHM**) as the one soliton solution, we find $\left(\frac{A_1^2 b_1^2}{\kappa_1} \sqrt{\frac{\pi}{2}}\right) + \left(\frac{A_2^2 b_2^2}{\kappa_2} \sqrt{\frac{\pi}{2}}\right) = 4$.

In general, approximating the eigenmode of eq.(1) with the dispersion map $\sigma(Z)$ by a Gaussian profile leads to errors in the reduced dynamics. However these errors can be reduced by an appropriate choice of the parameter A_j . To extend the previous work of the single **NLS** provided by Smith *et al.*⁵, we define the modified amplitudes-enhancement factor A_j as follows

$$A_j^2 = 1 + \alpha_j \left\{ \frac{\lambda_0^2}{2\pi c T_0^2} [(D_+ - \overline{D}) Z_+ - (D_- - \overline{D}) Z_-] \right\}^2, \quad j = 1, 2 \quad (14)$$

The amplitudes-enhancement factor $\alpha_j \neq 0$, if we take an arbitrary $A_1^2 \neq A_2^2$ or $(A_1^2 = A_2^2) \neq 1$ in order to characterize a more complicated the Gaussian pulses dynamics. This problem can also be easily discussed using a numerical computation and simulation. However, in order to simplify our analytical calculation we choose $A_1 = A_2 = A$ and then find $\kappa_1 = \kappa_2 = \frac{1.18}{1.76}$ if $b_2^2 = b_1^2 \approx 1.0702$.

After substituting the pulses ansatz of eq.(12) in the Lagrangian equations of eqs. (9) and (10) and evaluating the Lagrangian, we then find

$$\begin{aligned} L = & \sum_{j=1}^2 \frac{A_j^2 b_j^2}{\kappa_j} \left\{ \frac{\sqrt{\frac{\pi}{2}}}{2\eta_j^2} \frac{\partial \beta_j}{\partial Z} + \left(\sqrt{\frac{\pi}{2}} \right) \frac{\partial \phi_j(Z)}{\partial Z} - \frac{\Upsilon(Z) \sqrt{\pi} \eta_j}{2} (A_j^2 b_j^2) \right\} \\ & + \sum_{j=1}^2 \frac{A_j^2 b_j^2}{\kappa_j} \left\{ \sqrt{\frac{\pi}{2}} \sigma_j(Z) \kappa_j^2 \left[\eta_j^2 + \left(\frac{\beta_j^2}{\eta_j^2} \right) \right] \right\} \\ & - 2\Upsilon(Z) \eta_1 \eta_2 A_1^2 A_2^2 b_1^2 b_2^2 \sqrt{\frac{\pi}{2}} (\exp[(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2)]) \\ & - \left\{ \frac{i2\sqrt{\pi}}{\sqrt{\Upsilon(Z)}} (r_{a_j} \Omega_j) \right\}, \end{aligned} \quad (15)$$

where we have chosen

$$\text{Re} \left((\kappa_j \eta_j)^2 \right) > 0$$

or

$$\text{Re} \left((\kappa_j \eta_j)^2 \right) = 0 \quad \text{and} \quad \text{Im} \left((\kappa_j \eta_j)^2 \right) > 0,$$

and found

$$\Omega_j = \sum_{j=1}^2 \frac{A_j^2 b_j^2}{\kappa_j} (\omega_{1j} - \omega_{2j}), \quad (16)$$

where

$$(\omega_{1_j} - \omega_{2_j}) = \exp\left(\frac{-\beta_j^2 \kappa_j^2}{4\eta_j^2}\right) [\exp(i\phi_j(Z)) - 1] \quad (17)$$

Applying the variational method with the Gaussian trial function of eq.(12), we obtain that the related parameters of \mathbb{T}_{0_j} , and f_i in the noise terms are very small and equal to zero.

Although the complicated variations with respect to the free parameters η_j , and β_j can also yield the appropriate ordinary differential equations (**ODEs**), we will extend the Lagrangian formulation of the perturbative coupled **NLS** and will then try to develop the Hamiltonian formulation of these equations by means of a Legendre transformation. This method reveals a conserved quantity, which is utilized in the solution of the nonintegrable coupled **NLS** equation. The validity of the variational method and its associated dynamics are also in order related to the certain Gaussian pulse ansatz such as provided in eq.(12). Then, we can produce the correct pulses evolution. This certainly is not true in the case for which self-phases modulation dominates dispersion and temporal side lobes are generated, i.e., if A_j are too large.

Although Kutz *et al.* had already said that the variational method was **incapable** of describing or capturing the slight growth of continuous spectra, which is generated by the dispersion-management perturbations. We improve that it is possible if we can reduce all of the additional parameters such as the frequencies $f_j(Z)$ and center position of the pulses $\mathbb{T}_{0_j}(Z)$ to be identical with zero.

IV. The Hamiltonian Dynamics of The Coupled NLS

There are various ways and manners in deriving the solution and explanation of the amplitudes and chirps of the nonintegrable coupled **NLS**. However for those cases in the nonintegrable systems, almost all of the methods cannot work to derive the solutions. So, the only useful method we are going to use is the Hamiltonian formulation of the dynamics. The Hamiltonian system for the system is useful since solutions can be derived and represented in terms of a modified quadrature. The Lagrangian system of the preceding section is brought to Hamiltonian from via a Legendre transformation. We can directly define the related canonical coordinates β_j with its conjugate variables p_{β_j} as follows

$$p_{\beta_j} = \frac{\partial L}{\partial \left(\frac{d\beta_j}{dZ}\right)} = \frac{A_j^2}{\eta_j^2}, \quad j = 1, 2 \quad (18)$$

which represents the canonical momentums. The Legendre transformation defines the Hamiltonian

$$H(\beta_j, p_{\beta_j}) = p_{\beta_j} \frac{d\beta_j}{dZ} - L(\beta_j, p_{\beta_j}), \quad (19)$$

where we must change the Lagrangian in eq.(15) into the Lagrangian in the new canonical variables β_j and p_{β_j} and then substitute it into eq.(19), we obtain

$$\begin{aligned}
H(\beta_j, p_{\beta_j}) &= \sum_{j=1}^2 \left(p_{\beta_j} \frac{d\beta_j}{dZ} - p_{\beta_j} b_j^2 \sqrt{\frac{\pi}{2}} \sigma_j(Z) \kappa_j [\eta_j^4 + \beta_j^2] \right) \\
&\quad - \sum_{j=1}^2 \frac{p_{\beta_j} b_j^2 \sqrt{\frac{\pi}{2}}}{\kappa_j} \left\{ \frac{1}{2} \frac{d\beta_j}{dZ} + \eta_j^2 \frac{d\phi_j(Z)}{dZ} - \frac{\Upsilon(Z) \eta_j^5 b_j^2}{\sqrt{2}} (p_{\beta_j}) \right\} \\
&\quad + 2\sqrt{\frac{\pi}{2}} \Upsilon(Z) (p_{\beta_1}) (p_{\beta_2}) \eta_1^3 \eta_2^3 b_1^2 b_2^2 \exp(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2) \\
&\quad + \left(\frac{i2\sqrt{\pi}}{\sqrt{\Upsilon(Z)}} \right) \{ r_{a_1} \Omega_1(\beta_1, p_{\beta_1}) + r_{a_2} \Omega_2(\beta_2, p_{\beta_2}) \}, \quad (20)
\end{aligned}$$

where $\Omega_j(\beta_j, p_{\beta_j})$ are defined as follows

$$\Omega_j(\beta_j, p_{\beta_j}) = \frac{p_{\beta_j} b_j^2 \eta_j^2}{\kappa_j} \exp\left(\frac{-\beta_j^2 \kappa_j^2}{4\eta_j^2}\right) [\exp(i\phi_j(Z)) - 1]. \quad (21)$$

The Hamiltonian can then be calculated with eq.(19), and the evolution equations for η_j dan β_j are determined from the following Hamilton's equations

$$\frac{d\beta_j}{dZ} = \frac{\partial H(\beta_j, p_{\beta_j})}{\partial p_{\beta_j}}, \quad \frac{dp_{\beta_j}}{dZ} = -\frac{\partial H(\beta_j, p_{\beta_j})}{\partial \beta_j}. \quad (22)$$

In terms of η_j , we can evaluate the evolution equations in eq.(22) to be four **ODEs** stated the dynamical system of the Gaussian pulses as follows

$$\begin{aligned}
\frac{\partial \beta_1}{\partial Z} &= 2\sigma_1(Z) \kappa_1^2 [\eta_1^4 - \beta_1^2] - \frac{1}{\sqrt{2}} \Upsilon(Z) \eta_1^3 A_1^2 b_1^2 \\
&\quad - 2\Upsilon(Z) \eta_2 \eta_1^3 A_2^2 b_2^2 \kappa_1 (1 + 2\eta_1^2 \kappa_1^2) \exp(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2) \\
&\quad - \left(\frac{\kappa_1}{A_1^2 b_1^2} \right) \frac{i2\sqrt{2}r_{a_1}}{\sqrt{\Upsilon(Z)}} \left\{ \frac{\partial}{\partial \eta_1} \Omega_1 \right\}, \quad (23a)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \beta_2}{\partial Z} &= 2\sigma_2(Z) \kappa_2^2 [\eta_2^4 - \beta_2^2] - \frac{\Upsilon(Z)}{\sqrt{2}} \eta_2^3 A_2^2 b_2^2 \\
&\quad - 2\Upsilon(Z) \eta_1 \eta_2^3 A_1^2 b_1^2 \kappa_2 (1 + 2\eta_2^2 \kappa_2^2) \exp(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2) \\
&\quad - \left(\frac{\kappa_2}{A_2^2 b_2^2} \right) \frac{i2\sqrt{2}r_{a_2}}{\sqrt{\Upsilon(Z)}} \left\{ \frac{\partial}{\partial \eta_2} \Omega_2 \right\}, \quad (23b)
\end{aligned}$$

and

$$\begin{aligned} \frac{d\eta_j}{dZ} = & -b_j^2 \sqrt{\frac{\pi}{2}} \sigma_j(Z) \kappa_j \beta_j \eta_j \\ & + \frac{\eta_j^3}{A_j^2} \frac{i\sqrt{\pi} r_{a_j}}{\sqrt{\Upsilon(Z)}} \left(\frac{\partial}{\partial \beta_i} \Omega_j(\beta_j, p_{\beta_j}) \right), \quad j = 1, 2 \end{aligned} \quad (23c)$$

where

$$\frac{\partial}{\partial \eta_j} \Omega_j = \frac{\beta_j^2 \kappa_j A_j^2 b_j^2}{2\eta_j^3} \left(e^{(i\phi_j(Z))} - 1 \right) e^{\frac{-(\beta_j \kappa_j)^2}{4\eta_j^2}} \quad (24a)$$

and

$$\frac{\partial}{\partial \beta_j} \Omega_j = \frac{-\beta_j \kappa_j A_j^2 b_j^2}{2\eta_j^2} \left(e^{(i\phi_j(Z))} - 1 \right) e^{\frac{-(\beta_j \kappa_j)^2}{4\eta_j^2}} \quad (24b)$$

The results of the **ODEs** in eqs.(23) describe that the coupled pulses dynamics depends on the chirps interactions contributed to the pulses amplitudes. Loss and gain fluctuations related to the center position of the pulses and the frequencies of noises are set very small to this formalism. Hence, the system of equations describes the nonlinear enhancement amplitudes and width fluctuations along with the modified quadratic-chirp variations as the pulses propagate

through a given dispersion map. Note that the resulting **ODEs** system is identical to that derived by taking the variations

$$\frac{\delta L}{\delta \eta_j} = 0 \quad \text{and} \quad \frac{\delta L}{\delta \beta_j} = 0. \quad (25)$$

Since the resulting nonlinear **ODEs** system of eqs.(23) is the Hamiltonian, and $\sigma_j(Z)$ is piecewise constant, we can utilize the fact that

$$\frac{dH}{dZ} = 0 \quad (26)$$

and construct level sets of a scaled version of H rewritten in terms of η_j and

β_j ,

$$\begin{aligned}
C = & \left(\frac{\kappa_1}{b_1^2 \sqrt{\frac{\pi}{2}}} - \frac{1}{2} \right) \frac{1}{\eta_1^2} \frac{d\beta_1}{dZ} + \left(\frac{\kappa_1 A_2^2}{A_1^2 b_1^2 \sqrt{\frac{\pi}{2}}} - \frac{A_2^2 \frac{b_2^2}{\kappa_2}}{2 A_1^2 \frac{b_1^2}{\kappa_1}} \right) \frac{1}{\eta_2^2} \frac{d\beta_2}{dZ} \\
& - \left\{ \frac{d\phi_1(Z)}{dZ} + \frac{A_2^2 \frac{b_2^2}{\kappa_2}}{A_1^2 \frac{b_1^2}{\kappa_1}} \frac{d\phi_2(Z)}{dZ} \right\} - \sigma_1(Z) \kappa_1^2 \left[\eta_1^2 + \frac{\beta_1^2}{\eta_1^2} \right] \\
& - \frac{A_2^2 b_2^2 \kappa_1}{A_1^2 b_1^2} \sigma_2(Z) \kappa_2 \left[\eta_2^2 + \frac{\beta_2^2}{\eta_2^2} \right] \\
& + \left(\frac{\Upsilon(Z)}{\sqrt{2}} \right) (\eta_1 A_1^2 b_1^2) + \frac{A_2^4 \frac{b_2^2}{\kappa_2}}{A_1^2 \frac{b_1^2}{\kappa_1}} \left(\frac{\Upsilon(Z)}{\sqrt{2}} \right) (\eta_2 b_2^2) \\
& + 2\Upsilon(Z) A_2^2 \kappa_1 \eta_1 \eta_2 b_2^2 (\exp[(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2)]) \\
& + \frac{1}{A_1^2 b_1^2} \left(\frac{i2\sqrt{2}\kappa_1}{\sqrt{\Upsilon(Z)}} \right) \{ r_{a1} \Omega_1(\beta_1, p_{\beta_1}) + r_{a2} \Omega_2(\beta_1, p_{\beta_1}) \},
\end{aligned} \tag{27}$$

In order to reduce the eq.(27), we choose

$$b_2 = \left(\frac{2\kappa_2}{\sqrt{\frac{\pi}{2}}} \right)^{\frac{1}{2}}, \quad \text{and} \quad b_1 = \left(\frac{(4 - 2A_2^2) \kappa_1}{A_1^2 \sqrt{\frac{\pi}{2}}} \right)^{\frac{1}{2}}. \tag{28a}$$

The eq.(27) can be more simplified by choosing

$$b_j^2 = 2\sqrt{\frac{2}{\pi}} \kappa_j = b^2 \approx 1.0702, \tag{28b}$$

and $(\kappa_1 = \kappa_2 = \kappa = \frac{1.18}{1.76})$

$$A_1^2 = A_2^2 = 1. \tag{28c}$$

We then find the general solution of $\beta_1(\eta_1, \eta_2, \beta_2)$ as follows

$$\beta_1 = \pm \eta_1 \left\{ \gamma_1 [\eta_1 + \eta_2] - \gamma_2 - \gamma_3 \left(\eta_2^2 + \frac{\beta_2^2}{\eta_2^2} \right) + \gamma_4 \eta_1 \eta_2 e^{[\kappa^2(\eta_1^2 + \eta_2^2)]} + \gamma_5 - \eta_1^2 \right\}^{\frac{1}{2}}, \tag{29}$$

where

$$\gamma_1 = \left(\frac{2\Upsilon(Z)}{\sigma(Z) \kappa \sqrt{\pi}} \right), \tag{30a}$$

$$\gamma_2 = \left(\frac{C + \frac{d\phi_1(Z)}{dZ} + \frac{d\phi_2(Z)}{dZ}}{\sigma_1(Z) \kappa^2} \right), \tag{30b}$$

$$\gamma_3 = 1 \quad (\text{for } \sigma_1(Z) = \sigma_2(Z) = \sigma), \tag{30c}$$

$$\gamma_4 = \frac{4\sqrt{2}\Upsilon(Z)}{\sigma(Z) \sqrt{\pi}}, \tag{30d}$$

and

$$\gamma_5 = \left(\frac{i\sqrt{\pi}}{\sigma(Z)\kappa^2\sqrt{\Upsilon(Z)}} \right) [r_{a1}\Omega_1(\beta_1, p_{\beta_1}) + r_{a2}\Omega_2(\beta_2, p_{\beta_2})] \quad (30e)$$

where $\Omega_1(\beta_1, p_{\beta_1})$ and $\Omega_2(\beta_2, p_{\beta_2})$ are defined in eq.(21).

V. Phases-Plane Dynamics of The Perturbed Coupled NLS

It is often convenient and insightful to plot the phase-plane dynamics of a planar dynamical system such as eqs.(23). This geometric representation of the solution treats the distance Z as a parameter while plotting the amplitudes η_j and chirps β_j variables on the x and y axes, respectively. It elucidates some fundamental aspects of the coupled breathers dynamics, which will be clearly shown in what follows.

The phases-plane dynamics are markedly different depending on the sign of the dispersion. Thus we consider separately the anomalous- and normal-dispersion regimes. Before seeking the fixed points of eqs.(23), we will simplify the very complicated equations in order to get the phases-plane dynamics consisted of $\beta_1(\eta_1)$, $\beta_1(\eta_2)$, $\beta_2(\eta_1)$ and $\beta_2(\eta_1)$. This will easily work by choosing

$$\beta_2 = \rho_1(Z)\beta_1, \quad (31a)$$

and

$$\eta_2 = \rho_2(Z)\eta_1, \quad (31b)$$

or by setting $\rho_1(Z) = \rho_2(Z)$, we suggest and make our solution in the simplest forms:

$$\rho(Z) = \frac{\beta_2}{\beta_1} = \frac{\eta_2}{\eta_1} = \text{const.} \quad (31c)$$

After substituting eq.(31c) into eqs.(23) and eq.(29), we then find

$$\beta_1 = \pm\eta_1 \left\{ \gamma_a\eta_1 - \frac{\gamma_2}{\tau} - \gamma_b\eta_1^2 + \gamma_c\eta_1^2 \exp[\kappa^2\eta_1^2(1+\rho^2)] + \frac{\gamma_5}{\tau} - \frac{\eta_1^2}{\tau} \right\}^{\frac{1}{2}}, \quad (32a)$$

$$\beta_1 = \pm\eta_2 \left\{ \frac{\gamma_a}{\rho^3}\eta_2 - \frac{\gamma_2}{\rho^2\tau} - \frac{\gamma_b}{\rho^4}\eta_2^2 + \frac{\gamma_c}{\rho^4}\eta_2^2 \left(e^{\kappa^2(1+\frac{1}{\rho^2})\eta_2^2} \right) + \frac{\gamma_5}{\rho^2\tau} - \frac{1}{\rho^4\tau}\eta_2^2 \right\}^{\frac{1}{2}}, \quad (32b)$$

$$\beta_2 = \pm \eta_2 \left\{ \frac{\gamma_a \eta_2}{\rho} - \frac{\gamma_2}{\tau} - \frac{\gamma_b \eta_2^2}{\rho^2} + \frac{\gamma_c \eta_2^2 \left(e^{\kappa^2 \left(1 + \frac{1}{\rho^2} \right) \eta_2^2} \right)}{\rho^2} + \frac{\gamma_5}{\tau} - \frac{\eta_2^2}{\rho^2 \tau} \right\}^{\frac{1}{2}}, \quad (32c)$$

$$\beta_2 = \pm \eta_1 \left\{ \rho^2 \gamma_a \eta_1 - \frac{\rho^2 \gamma_2}{\tau} - \rho^2 \gamma_b \eta_1^2 + \rho^2 \gamma_c \eta_1^2 e^{\kappa^2 \eta_1^2 (1 + \rho^2)} + \frac{\rho^2 \gamma_5}{\tau} - \frac{\rho^2 \eta_1^2}{\tau} \right\}^{\frac{1}{2}}, \quad (32d)$$

where we have reduced the eq.(32) by choosing τ as follows ($\sigma_1(Z) = \sigma_2(Z)$)

$$\tau = 2, \quad (33)$$

and the parameters $\gamma_a, \gamma_b, \gamma_c$, and the revised γ_5 are

$$\gamma_a = \frac{\gamma_1 [1 + \rho(Z)]}{2}, \quad (34a)$$

$$\gamma_b = \frac{\gamma_3 \rho^2(Z)}{2}, \quad (34b)$$

$$\gamma_c = \frac{\gamma_4 \rho(Z)}{2}, \quad (34c)$$

and

$$\begin{aligned} \gamma_5 &= \left(\frac{i\sqrt{\pi} r_a}{\sigma(Z) \kappa^2 \sqrt{\Upsilon(Z)}} \right) [\Omega_1 + \Omega_2] \\ &= \frac{i\sqrt{\pi} r_a b^2 \Xi(Z)}{\sigma(Z) \kappa^3 \sqrt{\Upsilon(Z)}} \left[e^{(i\phi_1(Z) +)} + e^{(i\phi_2(Z))} - 2 \right], \text{ for } r_{a_1} = r_{a_2} = r_a, \end{aligned} \quad (34d)$$

or by choosing $r_{a_j} = \exp[-0.5i\phi_j(Z)]$ in the eq.(34d), we can reduce the imaginary and then find

$$\gamma_5 = \left(\frac{-2\sqrt{\pi} b^2 \Xi}{\sigma \kappa^3 \sqrt{\Upsilon}} \right) [\sin(0.5\phi_1) + \sin(0.5\phi_2)] \quad (34e)$$

where ($\Xi(Z) > 1$). In eq.(34d), we have suggested the following relationships

$$\Xi = - \frac{i8\sqrt{\Upsilon}\sigma\kappa}{\sqrt{\pi} r_a b^2 (e^{i\phi_1} + e^{i\phi_2} - 2)} \text{LambertW} \left(\frac{1}{8} i\sqrt{\pi} r_a \frac{b^2}{\sqrt{\Upsilon}} \frac{e^{i\phi_1} + e^{i\phi_2} - 2}{\sigma\kappa} (\mathbb{I}) \right), \quad (35a)$$

where

$$\mathcal{L} = \exp \left(-\frac{1}{4}\kappa^2\gamma_a\eta_1 - \frac{1}{4}\kappa^2\gamma_c\eta_1^2 e^{(\kappa^2\eta_1^2(1+\rho^2))} + \frac{1}{8}\kappa^2\gamma_2 + \frac{1}{4}\kappa^2\eta_1^2\gamma_b + \frac{1}{8}\kappa^2\eta_1^2 \right),$$

and

$$-\left(\frac{\kappa^2}{4\eta_1^2}\right)\beta_1^2 + \left(\frac{\kappa^2}{8\eta_1^2}\right)\beta_1^4 + O(\beta_1^6) > 0, \quad (35b)$$

and the rule appeared in eqs.(35a) and (35b) must be used to our entire solutions in this section. If we choose $\Xi(Z) = 1$ or the chirp $\beta_1 \approx 0$, this then means that the pulse width is minimum at the center of unit cell. However, the mean-square frequency shift is not minimum. We can also suppose that a noise (in γ_5 term) sourced by the amplifier which has a small frequency causes a small change in the coupled breathers dynamics of the Gaussian pulses. However, we still can reduce the noise by enhancing the amplitude of the pulses.

The aim of the use of a relationship in eq.(31c) is to make the phases-plane dynamics of the coupled breathers are in stable conditions. A first order **ODE** in terms of η_1 alone can then be derived by substituting eq. (32a) into eq.(23c):

$$\begin{aligned} \frac{d\eta_1}{dZ} &= \mp \left(b^2 \sqrt{\frac{\pi}{2}} \sigma_1(Z) \kappa \eta_1^2 \right) \\ &\times \left\{ \gamma_a \eta_1 - \frac{\gamma_2}{\tau} - \gamma_b \eta_1^2 + \gamma_c \eta_1^2 \exp [\kappa^2 \eta_1^2 (1 + \rho_2^2)] + \frac{\gamma_5}{\tau} - \frac{\eta_1^2}{\tau} \right\}^{\frac{1}{2}} \\ &\mp \left(\frac{i r_{a_1} \sqrt{\pi} \Xi(Z) (e^{(i\phi_1(Z))} - 1) \kappa b^2 \eta_1^2}{2\sqrt{\Upsilon(Z)}} \right) \\ &\times \left\{ \gamma_a \eta_1 - \frac{\gamma_2}{\tau} - \gamma_b \eta_1^2 + \gamma_c \eta_1^2 \exp [\kappa^2 \eta_1^2 (1 + \rho_2^2)] + \frac{\gamma_5}{\tau} - \frac{\eta_1^2}{\tau} \right\}^{\frac{1}{2}} \end{aligned} \quad (36)$$

Upon integrating this equation with initial conditions $\eta_1 = \eta_{01}$ at $Z = 0$, we find

$$Z = \int_{\eta_{01}}^{\eta_1} \frac{\mp \left\{ \gamma_a \eta_1 - \frac{\gamma_2}{\tau} - \gamma_b \eta_1^2 + \gamma_c \eta_1^2 \exp [\kappa^2 \eta_1^2 (1 + \rho_2^2)] + \frac{\gamma_5}{\tau} - \frac{\eta_1^2}{\tau} \right\}^{-\frac{1}{2}} d\eta_1}{\left\{ b^2 \sqrt{\frac{\pi}{2}} \sigma_1(Z) \kappa + \frac{i \sqrt{\pi} \Xi(Z) (e^{(i\phi_1(Z))} - 1) \kappa b^2 r_{a_1}}{2\sqrt{\Upsilon(Z)}} \right\} \eta_1^2} \quad (37)$$

Another first order **ODE** in terms of η_2 alone will easily be derived using the same manner. Although we already get an analytic solution for the pulse dynamics in eq.(37), the integration is still very complicated due to some closed-form expressions. The simplest way left to explain the coupled breathers dynamics is a geometrical interpretation of the **ODEs** in eqs.(23).

Finally, we come to obtain the fixed points of eqs.(23) by setting the right-hand sides equal to zero. There are three fixed points which can be proved by figuring the phases-plane dynamics consisted of $\beta_1(\eta_1)$, $\beta_1(\eta_2)$, $\beta_2(\eta_1)$ and $\beta_2(\eta_1)$:

$$\text{I. } \beta_j = 0, \quad \eta_j = 0, \quad (38a)$$

$$\begin{aligned} \text{II. } \beta_j &= 0, \quad \text{and} \\ \eta_1 &= \frac{a_{21}a_{71}}{a_{71}a_{11} - a_{41} \exp(\vartheta_1) a_{31}\vartheta_1 - a_{71} \exp(\vartheta_1) a_{31}} = \eta_{10}, \end{aligned} \quad (38b)$$

$$\begin{aligned} \text{III. } \beta_j &= 0, \quad \text{and} \\ \eta_2 &= \frac{c_{21}c_{71}}{c_{71}c_{11} - c_{41} \exp(\vartheta'_1) c_{31}\vartheta'_1 - c_{71} \exp(\vartheta'_1) c_{31}} = \eta_{20}, \end{aligned} \quad (38c)$$

where ϑ_1 is a root of

$$\begin{aligned} &\{e^{2\Theta} \Theta a_{71}^2 a_{31}^2 + 2e^{2\Theta} a_{71} a_{31}^2 a_{41} \Theta^2 + e^{2\Theta} a_{41}^2 a_{31}^2 \Theta^3 - 2e^{\Theta} \Theta a_{71}^2 a_{11} a_{31} \\ &- 2e^{\Theta} a_{71} a_{11} a_{41} a_{31} \Theta^2 - a_{21}^2 a_{71}^3 + \Theta a_{71}^2 a_{11}^2\}, \end{aligned} \quad (39a)$$

and ϑ'_1 is a root of

$$\begin{aligned} &\{e^{2\Theta} \Theta c_{71}^2 c_{31}^2 + 2e^{2\Theta} c_{71} c_{31}^2 c_{41} \Theta^2 + e^{2\Theta} c_{41}^2 c_{31}^2 \Theta^3 - 2e^{\Theta} \Theta c_{71}^2 c_{11} c_{31} \\ &- 2e^{\Theta} c_{71} c_{11} c_{41} c_{31} \Theta^2 - c_{21}^2 c_{71}^3 + \Theta c_{71}^2 c_{11}^2\}, \end{aligned} \quad (39b)$$

and the parameters $\eta_{01}, \eta_{02}, a_{11}, a_{21}, a_{31}, a_{41}, a_{71}, c_{11}, c_{21}, c_{31}, c_{41}$, and c_{71} (for $A_1 = A_2 = A$ and $\sigma_1(Z) = \sigma_2(Z) = \sigma(Z)$) are

$$\begin{aligned} \eta_{01} &= \frac{\frac{\Upsilon(Z)}{\sqrt{2}} A^2 b^2 \kappa^2 (1 + \rho^2)}{2\kappa^4 (1 + \rho^2) \sigma - 2\kappa^2 \exp(\vartheta_1) [2\Upsilon(Z) \rho A^2 b^2 \kappa] \vartheta_1 - \kappa^2 (1 + \rho^2) \exp(\vartheta_1) [2\Upsilon(Z) \rho A^2 b^2 \kappa]}, \\ \eta_{02} &= \frac{\frac{\Upsilon(Z)}{\sqrt{2}} A^2 b^2 \kappa^2 \left(1 + \frac{1}{\rho^2}\right)}{2\kappa^4 \left(1 + \frac{1}{\rho^2}\right) \sigma - 2\kappa^2 \exp(\vartheta'_1) [2\Upsilon(Z) \rho A^2 b^2 \kappa] \vartheta'_1 - \kappa^2 \left(1 + \frac{1}{\rho^2}\right) \exp(\vartheta'_1) [2\Upsilon(Z) \rho A^2 b^2 \kappa]}, \end{aligned}$$

$$\begin{aligned} a_{11} &= c_{11} = 2\sigma(Z) \kappa^2, \\ a_{21} &= c_{21} = \frac{A^2 \Upsilon(Z) b^2}{\sqrt{2}}, \\ a_{31} &= c_{31} = 2\Upsilon(Z) \rho A^2 b^2 \kappa, \\ a_{41} &= c_{41} = 2\kappa^2, \\ a_{71} &= \kappa^2 (1 + \rho^2), \\ c_{71} &= \kappa^2 \left(1 + \frac{1}{\rho^2}\right). \end{aligned}$$

A linear-stability calculation about each of the fixed points gives insight into the general coupled breathers dynamics of the system. In particular, the fixed points $(\eta_j, \beta_j) = (0, 0)$ are degenerate so that their linear stabilities cannot be determined (see, for examples, figures 1, 3, 5, and 7). However, the linear stability of the fixed point II and III can be determined by studying the pulses-plane dynamics of the **ODEs** in eq.(23). Linearizing about these points by setting

$$\eta_j = \eta_{0j} + \widetilde{\eta}_j \quad (40a)$$

$$\beta_j = 0 + \widetilde{\beta}_j \quad (40b)$$

yields the linear system and it can be rewritten in a matrix form as follows

$$\begin{pmatrix} \frac{d\widetilde{\eta}_1}{dZ} \\ \frac{d\widetilde{\eta}_2}{dZ} \\ \frac{\partial \widetilde{\beta}_1}{\partial Z} \\ \frac{\partial \widetilde{\beta}_2}{\partial Z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & M_{13} & 0 \\ 0 & 0 & 0 & M_{24} \\ M_{31} & 0 & 0 & 0 \\ 0 & M_{42} & 0 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\eta}_1 \\ \widetilde{\eta}_2 \\ \widetilde{\beta}_1 \\ \widetilde{\beta}_2 \end{pmatrix}, \quad (41)$$

where

$$\begin{aligned} M_{13} = & -b^2 \sqrt{\frac{\pi}{2}} \sigma_1(Z) \kappa \eta_{01} \\ & - \frac{ir_{a1} \sqrt{\pi} (e^{(i\phi_1(Z))} - 1) \kappa b^2 \Xi(Z) \eta_{01}}{2\sqrt{\Upsilon(Z)}} \end{aligned} \quad (42a)$$

$$\begin{aligned} M_{24} = & -b^2 \sqrt{\frac{\pi}{2}} \sigma_2(Z) \kappa \eta_{02} \\ & - \frac{ir_{a2} \sqrt{\pi} (e^{(i\phi_2(Z))} - 1) \kappa b^2 \Xi(Z) \eta_{02}}{2\sqrt{\Upsilon(Z)}} \end{aligned} \quad (42b)$$

$$\begin{aligned} M_{31} = & 8\sigma_1(Z) \kappa^2 \eta_{01}^3 - \frac{3}{\sqrt{2}} \Upsilon(Z) A_1^2 b^2 \eta_{01}^2 \\ & - 8\Upsilon(Z) \rho A_2^2 b^2 \kappa \eta_{01}^3 (1 + 2\eta_{01}^2 \kappa^2) e^{(\kappa^2(1+\rho^2)\eta_{01}^2)} \\ & - 8\Upsilon(Z) \rho A_2^2 b^2 \kappa^3 \eta_{01}^5 e^{(\kappa^2(1+\rho^2)\eta_{01}^2)} \\ & - 4\Upsilon(Z) \rho A_2^2 b^2 \kappa^3 \eta_{01}^5 (1 + 2\eta_{01}^2 \kappa^2) (1 + \rho^2) e^{(\kappa^2(1+\rho^2)\eta_{01}^2)} \end{aligned} \quad (42c)$$

$$\begin{aligned}
M_{42} = & 8\sigma_2(Z)\kappa^2\eta_{02}^3 - \frac{3\Upsilon(Z)}{\sqrt{2}}A_2^2b^2\eta_{02}^2 \\
& - \frac{8\Upsilon(Z)}{\rho}A_1^2b^2\kappa\eta_{02}^3(1+2\eta_{02}^2\kappa^2)e^{\left(\kappa^2\left(1+\frac{1}{\rho^2}\right)\eta_{02}^2\right)} \\
& - \frac{8\Upsilon(Z)}{\rho}A_1^2b^2\kappa^3\eta_{02}^5e^{\left(\kappa^2\left(1+\frac{1}{\rho^2}\right)\eta_{02}^2\right)} \\
& - \frac{4\Upsilon(Z)}{\rho}A_1^2b^2\kappa^3\eta_{02}^5(1+2\eta_{02}^2\kappa^2)\left(1+\frac{1}{\rho^2}\right)e^{\left(\kappa^2\left(1+\frac{1}{\rho^2}\right)\eta_{02}^2\right)}. \quad (42d)
\end{aligned}$$

The eigenvalues of this system (eq.(41)) determine the stability near each of the fixed points and are given by

$$\lambda_{1,2} = \pm\sqrt{(M_{31}M_{13})}, \quad (43a)$$

$$\lambda_{3,4} = \pm\sqrt{(M_{42}M_{24})} \quad (43b)$$

The eqs.(43) show that the fixed points **II** and **III** ($(\eta_1, \beta_1 = 0)$, $(\eta_2, \beta_1 = 0)$, $(\eta_1, \beta_2 = 0)$ and $(\eta_2, \beta_2 = 0)$) are the centers if the sign of the dispersion (σ) is positive. We will then investigate both the anomalous- and normal-dispersion fibers in what follows.

A. Normal Dispersion: $(\sigma_1 = \sigma_2 = \sigma) = \frac{D_-}{D} < 0$

We start by considering the phases-plane dynamics in the normal dispersion regime. In this case, we can rewrite $\sigma = -\frac{|D_-|}{D} < 0$. We then consider the location and stability of the three fixed points of eqs.(23) shown in eqs.(38). Fixed point **I** has already been determined to be generate, and fixed points **II** and **III** lay in the left half-plane since $\eta_{0j} < 0$. Because we consider only values of $\eta_j \geq 0$, only the fixed points at the origin are relevant. the shapes of the plane are the same as homoclinic orbits, which emanate and terminate in the origin.

B. Anomalous Dispersion: $(\sigma_1 = \sigma_2 = \sigma) = \frac{D_+}{D} > 0$

In the anomalous-dispersion regime, the phases-plane dynamics are significantly different than that of the normal-dispersion regime since fixed points **II** and **III**, given by eqs.(38b) and (38c) are located at

$$\beta_j = 0, \text{ and}$$

$$\begin{aligned}
\text{II.} \quad & \frac{\frac{\Upsilon(Z)}{\sqrt{2}}A^2b^2\kappa^2(1+\rho^2)}{\frac{2\kappa^4(1+\rho^2)D_+}{D} - 2\kappa^2\exp(\vartheta_1)[2\Upsilon(Z)\rho A^2b^2\kappa]\vartheta_1 - \kappa^2(1+\rho^2)\exp(\vartheta_1)[2\Upsilon(Z)\rho A^2b^2\kappa]} > 0, \quad (44a)
\end{aligned}$$

and

$$\beta_j = 0, \quad \text{and}$$

$$\text{III.} \quad \frac{\frac{\Upsilon(Z)}{\sqrt{2}} A^2 b^2 \kappa^2 \left(1 + \frac{1}{\rho^2}\right)}{\frac{2\kappa^4 \left(1 + \frac{1}{\rho^2}\right) D_+}{D} - 2\kappa^2 \exp(\vartheta'_1) [2\Upsilon(Z) \rho A^2 b^2 \kappa] \vartheta'_1 - \kappa^2 \left(1 + \frac{1}{\rho^2}\right) \exp(\vartheta'_1) [2\Upsilon(Z) \rho A^2 b^2 \kappa]} > 0, \quad (44b)$$

with the four eigenvalues (for $\sigma = \frac{D_+}{D}$)

$$\lambda_{1,2} = \pm \sqrt{(M_{31} M_{13})}, \quad (45a)$$

$$\lambda_{3,4} = \pm \sqrt{(M_{42} M_{24})} \quad (45b)$$

In this case, the phases flow for $\eta_j \geq 0$ are partially determined by the three critical points **I**, **II** and **III**. It can be more obviously understood if we replace $C = -\left(\frac{d\phi_1(Z)}{dZ} + \frac{d\phi_2(Z)}{dZ}\right)$ in the eqs.(32) and $r_{a_j} = \exp(-0.5i\phi_j(Z))$ in eq.(30e), We then give rise to the separatix

$$\beta_1 = \pm \eta_1 \left\{ \gamma_a \eta_1 + \gamma_c \eta_1^2 \exp[\kappa^2 \eta_1^2 (1 + \rho^2)] + \frac{\gamma_5}{2} - \left(\gamma_b + \frac{1}{2}\right) \eta_1^2 \right\}^{\frac{1}{2}}, \quad (46a)$$

$$\beta_1 = \pm \eta_2 \left\{ \frac{\gamma_a}{\rho^3} \eta_2 + \frac{\gamma_c}{\rho^4} \eta_2^2 \left(e^{\kappa^2 (1 + \frac{1}{\rho^2}) \eta_2^2} \right) + \frac{\gamma_5}{2\rho^2} - \left(\frac{1 + 2\gamma_b}{2\rho^4} \right) \eta_2^2 \right\}^{\frac{1}{2}}, \quad (46b)$$

$$\beta_2 = \pm \eta_2 \left\{ \frac{\gamma_a \eta_2}{\rho} + \frac{\gamma_c \eta_2^2 \left(e^{\kappa^2 (1 + \frac{1}{\rho^2}) \eta_2^2} \right)}{\rho^2} + \frac{\gamma_5}{2} - \left(\frac{1 + 2\gamma_b}{2\rho^2} \right) \eta_2^2 \right\}^{\frac{1}{2}}, \quad (46c)$$

and

$$\beta_2 = \pm \eta_1 \left\{ \rho^2 \gamma_a \eta_1 + \rho^2 \gamma_c \eta_1^2 \left(e^{\kappa^2 \eta_1^2 (1 + \rho^2)} \right) + \frac{\rho^2 \gamma_5}{2} - \left(\rho^2 \gamma_b + \frac{\rho^2}{2} \right) \eta_1^2 \right\}^{\frac{1}{2}}, \quad (46d)$$

which have a cusp at the origin ($\beta_j = \eta_j = 0$). The solutions outside these separatix eventually flow into the origin (homoclinic orbits), while those inside

the separatix are periodic. In eqs.(46), there is an indicator γ_5 represented the contribution of a perturbation term caused by a noise sourced by amplifiers in optical fiber and it can influence the amplitudes and chirps of the pulses dynamics. The geometrical figures and its analysis of this system of eqs.(23) are not provided in this paper due to a complicated interpretation. We will then show the simplest explanation of the unperturbed dynamical system in the following section.

VI. The Hamiltonian Dynamics of The Unperturbed Coupled NLS

The solution of β_1 in eq.(29) can be reduced to that in **the single NLS** which has no the perturbation terms provided by Kutz, et al. by setting $\beta_2 = \eta_2 = \phi_2 = 0$, $\Upsilon(Z) = 1$ and $\epsilon_j R_j = 0$. On the other hand, we conclude that the terms of the Lagrangian, the **ODEs** and the β solutions of the unperturbed coupled **NLS** :

$$\begin{aligned} i\frac{\partial U_1}{\partial Z} + \frac{\sigma(Z)}{2}\frac{\partial^2 U_1}{\partial T^2} + \left(|U_1|^2 + |U_2|^2\right)U_1 &= 0 \\ i\frac{\partial U_2}{\partial Z} + \frac{\sigma(Z)}{2}\frac{\partial^2 U_2}{\partial T^2} + \left(|U_1|^2 + |U_2|^2\right)U_2 &= 0 \end{aligned} \quad (47)$$

reduced from the results in section 5 are, respectively, as follows

$$\begin{aligned} L = & \sum_{j=1}^2 \frac{A_j^2 b_j^2}{\kappa_j} \left\{ \frac{\sqrt{\pi}}{2} \frac{\partial \beta_j}{\partial Z} + \left(\sqrt{\frac{\pi}{2}} \right) \frac{\partial \phi_j(Z)}{\partial Z} - \frac{\sqrt{\pi} \eta_j}{2} (A_j^2 b_j^2) \right\} \\ & + \sum_{j=1}^2 \frac{A_j^2 b_j^2}{\kappa_j} \left\{ \sqrt{\frac{\pi}{2}} \sigma_j(Z) \kappa_j^2 \left[\eta_j^2 + \left(\frac{\beta_j^2}{\eta_j^2} \right) \right] \right\} \\ & - 2\sqrt{\frac{\pi}{2}} \eta_1 \eta_2 A_1^2 A_2^2 b_1^2 b_2^2 \left(\exp \left[(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2) \right] \right), \end{aligned} \quad (48a)$$

$$\begin{aligned} \frac{\partial \beta_1}{\partial Z} = & 2\sigma_1(Z) \kappa_1^2 [\eta_1^4 - \beta_1^2] - \frac{1}{\sqrt{2}} \eta_1^3 A_1^2 b_1^2 \\ & - 2\eta_2 \eta_1^3 A_2^2 b_2^2 \kappa_1 (1 + 2\eta_1^2 \kappa_1^2) \exp(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2) \end{aligned} \quad (48b)$$

$$\begin{aligned} \frac{\partial \beta_2}{\partial Z} = & 2\sigma_2(Z) \kappa_2^2 [\eta_2^4 - \beta_2^2] - \frac{1}{\sqrt{2}} \eta_2^3 A_2^2 b_2^2 \\ & - 2\eta_1 \eta_2^3 A_1^2 b_1^2 \kappa_2 (1 + 2\eta_2^2 \kappa_2^2) \exp(\kappa_1^2 \eta_1^2 + \kappa_2^2 \eta_2^2), \end{aligned} \quad (48c)$$

$$\frac{d\eta_j}{dZ} = -b_j^2 \sqrt{\frac{\pi}{2}} \sigma_j(Z) \kappa_j \beta_j \eta_j, \quad j = 1, 2 \quad (48d)$$

and

$$\beta_1 = \pm \eta_1 \left\{ \gamma_1 [\eta_1 + \eta_2] - \gamma_2 - \gamma_3 \left(\eta_2^2 + \frac{\beta_2^2}{\eta_2^2} \right) + \gamma_4 \eta_1 \eta_2 e^{[\kappa^2(\eta_1^2 + \eta_2^2)]} - \eta_1^2 \right\}^{\frac{1}{2}}. \quad (49)$$

The simplest stable solution of the equations for the amplitudes η_j and chirps β_j derived from eq.(49) will be found by choosing $\eta_2 = \rho \eta_1$, and $\beta_2 = \rho \beta_1$,

$$\beta_1 = \pm \frac{\eta_1}{\sqrt{2}} \left\{ \gamma_1 [1 + \rho] \eta_1 - \gamma_2 - \rho^2 \eta_1^2 + \gamma_4 \rho \eta_1^2 e^{[\kappa^2(1+\rho^2)\eta_1^2]} - \eta_1^2 \right\}^{\frac{1}{2}}, \quad (50a)$$

$$\beta_1 = \pm \frac{\eta_2}{\rho \sqrt{2}} \left\{ \gamma_1 [1 + \rho] \frac{\eta_2}{\rho} - \gamma_2 - \eta_2^2 + \gamma_4 \frac{\eta_2^2}{\rho} e^{\left[\frac{\kappa^2(1+\rho^2)}{\rho^2} \eta_2^2 \right]} - \frac{\eta_2^2}{\rho^2} \right\}^{\frac{1}{2}}, \quad (50b)$$

$$\beta_2 = \pm \left(\frac{\rho}{\sqrt{2}} \right) \eta_1 \left\{ \gamma_1 [1 + \rho] \eta_1 - \gamma_2 - \rho^2 \eta_1^2 + \gamma_4 \rho \eta_1^2 e^{[\kappa^2(1+\rho^2)\eta_1^2]} - \eta_1^2 \right\}^{\frac{1}{2}}, \quad (50c)$$

and

$$\beta_2 = \pm \frac{\eta_2}{\sqrt{2}} \left\{ \gamma_1 [1 + \rho] \frac{\eta_2}{\rho} - \gamma_2 - \eta_2^2 + \gamma_4 \frac{\eta_2^2}{\rho} e^{\left[\frac{\kappa^2(1+\rho^2)}{\rho^2} \eta_2^2 \right]} - \frac{\eta_2^2}{\rho^2} \right\}^{\frac{1}{2}}, \quad (50d)$$

where (for $\sigma_1(Z) = \sigma_2(Z) = \sigma(Z)$ and $b_1 = b_2 = b$)

$$\gamma_1 = \left(\frac{2}{\sigma(Z) \kappa \sqrt{\pi}} \right), \quad (51a)$$

$$\gamma_2 = \left(\frac{C + \frac{d\phi_1(Z)}{dZ} + \frac{d\phi_2(Z)}{dZ}}{\sigma(Z) \kappa^2} \right), \quad (51b)$$

$$\gamma_4 = \frac{4\sqrt{2}}{\sigma(Z) \sqrt{\pi}}. \quad (51c)$$

The substitution of eqs.(50a) and (50d) into eq.(48d) yields the following first **ODEs** in terms of η_1 and η_2 , respectively:

$$\frac{d\eta_1}{dZ} = \mp \frac{b^2 \sqrt{\pi} \sigma(Z) \kappa \eta_1^2}{2} \left\{ \gamma_1 [1 + \rho] \eta_1 - \gamma_2 - \rho^2 \eta_1^2 + \gamma_4 \rho \eta_1^2 e^{[\kappa^2(1+\rho^2)\eta_1^2]} - \eta_1^2 \right\}^{\frac{1}{2}}, \quad (52a)$$

and

$$\frac{d\eta_2}{dZ} = \mp \frac{b^2 \sqrt{\pi} \sigma(Z) \kappa \eta_2^2}{2} \left\{ \gamma_1 [1 + \rho] \frac{\eta_2}{\rho} - \gamma_2 - \eta_2^2 + \frac{\gamma_4 \eta_2^2}{\rho} e^{\left[\frac{\kappa^2(1+\rho^2)}{\rho^2} \eta_2^2 \right]} - \frac{\eta_2^2}{\rho^2} \right\}^{\frac{1}{2}}. \quad (52b)$$

Upon integrating these equations with initial conditions $\eta_j = \eta_{0_j}$ at $Z = 0$, we find

$$Z = \mp 2 \int_{\eta_{01}}^{\eta_1} \frac{\left\{ \gamma_1 [1 + \rho] \eta_1 - \gamma_2 - \rho^2 \eta_1^2 + \gamma_4 \rho \eta_1^2 e^{\left[\kappa^2 (1 + \rho^2) \eta_1^2 \right]} - \eta_1^2 \right\}^{-\frac{1}{2}} d\eta_1}{b^2 \sqrt{\pi} \sigma(Z) \kappa \eta_1^2}, \quad (53a)$$

or

$$Z = \mp 2 \int_{\eta_{02}}^{\eta_2} \frac{\left\{ \gamma_1 \left[1 + \frac{1}{\rho} \right] \eta_2 - \gamma_2 - \eta_2^2 + \gamma_4 \frac{\eta_2^2}{\rho} \exp \left[\frac{\kappa^2 (1 + \rho^2) \eta_2^2}{\rho^2} \right] - \frac{\eta_2^2}{\rho^2} \right\}^{-\frac{1}{2}} d\eta_2}{b^2 \sqrt{\pi} \sigma(Z) \kappa \eta_2^2}, \quad (53b)$$

where $\sigma(Z)$ is a fixed value, i.e., $\sigma_1 = \sigma_2 = \frac{D_-}{D}$ or $\frac{D_+}{D}$, depending on whether the coupled pulses are propagating in the normal or anomalous regime, respectively. The integrations in eq.(53) are very complicated expressions, which must be transformed to give $\eta_j = \eta_j(Z)$. However, we will explain it elsewhere. The only simplest manner in describing the coupled breathers dynamics is by using a geometrical interpretation of eqs.(48b), (48c) and (48d).

The fixed points derived from a perturbative coupled **NLS** are usually very complicated in terms of the roots. However, in order to explain the phases-plane dynamics of the coupled breathers, we come to obtain the unperturbed fixed points of eqs.(48) by setting the right-hand sides equal to zero. Here, we find three fixed points reduced from that in eqs.(44):

$$\text{I.} \quad \beta_j = 0, \quad \eta_j = 0, \quad (54a)$$

$$\begin{aligned} \text{II.} \quad \beta_j &= 0, \\ \eta_1 &= \frac{a_{21} a_{71}}{a_{71} a_{11} - a_{41} \exp(\vartheta_1) a_{31} \vartheta_1 - a_{71} \exp(\vartheta_1) a_{31}} = \eta_{10}, \end{aligned} \quad (54b)$$

$$\begin{aligned} \text{III.} \quad \beta_j &= 0, \\ \eta_2 &= \frac{c_{21} c_{71}}{c_{71} c_{11} - c_{41} \exp(\vartheta'_1) c_{31} \vartheta'_1 - c_{71} \exp(\vartheta'_1) c_{31}} = \eta_{20}, \end{aligned} \quad (54c)$$

where ϑ_1 is a root of

$$\{e^{2\Theta}\Theta a_{71}^2 a_{31}^2 + 2e^{2\Theta} a_{71} a_{31}^2 a_{41} \Theta^2 + e^{2\Theta} a_{41}^2 a_{31}^2 \Theta^3 - 2e^{\Theta}\Theta a_{71}^2 a_{11} a_{31} - 2e^{\Theta} a_{71} a_{11} a_{41} a_{31} \Theta^2 - a_{21}^2 a_{71}^3 + \Theta a_{71}^2 a_{11}^2\}, \quad (55a)$$

and ϑ'_1 is a root of

$$\{e^{2\Theta}\Theta c_{71}^2 c_{31}^2 + 2e^{2\Theta} c_{71} c_{31}^2 c_{41} \Theta^2 + e^{2\Theta} c_{41}^2 c_{31}^2 \Theta^3 - 2e^{\Theta}\Theta c_{71}^2 c_{11} c_{31} - 2e^{\Theta} c_{71} c_{11} c_{41} c_{31} \Theta^2 - c_{21}^2 c_{71}^3 + \Theta c_{71}^2 c_{11}^2\}, \quad (55b)$$

in which the roots ϑ_1 and ϑ'_1 are, respectively, $\vartheta_1 = 1.3165$, and $\vartheta'_1 = 6.189 + 3.136i$ if $A = 1$, $\sigma = 1$, and $\rho = 0.03$, and the parameters $\eta_{01}, \eta_{02}, a_{11}, a_{21}, a_{31}, a_{41}, a_{71}, c_{11}, c_{21}, c_{31}, c_{41}$, and c_{71} ($A_1 = A_2 = A$) are

$$\eta_{01} = \frac{\frac{1}{\sqrt{2}} A^2 b^2 \kappa^2 (1 + \rho^2)}{2\kappa^4 (1 + \rho^2) \sigma - 2\kappa^2 \exp(\vartheta_1) [2\rho A^2 b^2 \kappa] \vartheta_1 - \kappa^2 (1 + \rho^2) \exp(\vartheta_1) [2\rho A^2 b^2 \kappa]}, \quad (56a)$$

$$\eta_{02} = \frac{\frac{1}{\sqrt{2}} A^2 b^2 \kappa^2 \left(1 + \frac{1}{\rho^2}\right)}{2\kappa^4 \left(1 + \frac{1}{\rho^2}\right) \sigma - 2\kappa^2 \exp(\vartheta'_1) [2\rho A^2 b^2 \kappa] \vartheta'_1 - \kappa^2 \left(1 + \frac{1}{\rho^2}\right) \exp(\vartheta'_1) [2\rho A^2 b^2 \kappa]}, \quad (56b)$$

$$\begin{aligned} a_{11} &= c_{11} = 2\sigma(Z) \kappa^2, \\ a_{21} &= c_{21} = \frac{A^2 b^2}{\sqrt{2}}, \\ a_{31} &= c_{31} = 2\rho A^2 b^2 \kappa, \\ a_{41} &= c_{41} = 2\kappa^2, \\ a_{71} &= \kappa^2 (1 + \rho^2), \\ c_{71} &= \kappa^2 \left(1 + \frac{1}{\rho^2}\right). \end{aligned}$$

Linearizing of eqs.(48b), (48c) and (48d) by setting $\eta_j = \eta_{0j} + \tilde{\eta}_j$ and $\beta_j = 0 + \tilde{\beta}_j$, we find the linear system by replacing $\epsilon_j R_j = 0$ or $r_{a_1} = 0$, and $\Upsilon(Z) = 1$ in eqs.(42) and it can be rewritten in a matrix form as follows

$$\begin{pmatrix} \frac{d\tilde{\eta}_1}{dZ} \\ \frac{d\tilde{\eta}_2}{dZ} \\ \frac{\partial\tilde{\beta}_1}{\partial Z} \\ \frac{\partial\tilde{\beta}_2}{\partial Z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & M'_{13} & 0 \\ 0 & 0 & 0 & M'_{24} \\ M'_{31} & 0 & 0 & 0 \\ 0 & M'_{42} & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \tilde{\beta}_1 \\ \tilde{\beta}_2 \end{pmatrix}, \quad (57)$$

where

$$M'_{13} = -b^2 \sqrt{\frac{\pi}{2}} \sigma(Z) \kappa \eta_{01} \quad (58a)$$

$$M'_{24} = -b^2 \sqrt{\frac{\pi}{2}} \sigma(Z) \kappa \eta_{02} \quad (58b)$$

$$\begin{aligned} M'_{31} = & 8\sigma(Z) \kappa^2 \eta_{01}^3 - \frac{3}{\sqrt{2}} A^2 b^2 \eta_{01}^2 \\ & - 8\rho A^2 b^2 \kappa \eta_{01}^3 (1 + 2\eta_{01}^2 \kappa^2) e^{(\kappa^2(1+\rho^2)\eta_{01}^2)} \\ & - 8\rho A^2 b^2 \kappa^3 \eta_{01}^5 e^{(\kappa^2(1+\rho^2)\eta_{01}^2)} \\ & - 4\rho A^2 b^2 \kappa^3 \eta_{01}^5 (1 + 2\eta_{01}^2 \kappa^2) (1 + \rho^2) e^{(\kappa^2(1+\rho^2)\eta_{01}^2)} \end{aligned} \quad (58c)$$

$$\begin{aligned} M'_{42} = & 8\sigma(Z) \kappa^2 \eta_{02}^3 - \frac{3}{\sqrt{2}} A^2 b^2 \eta_{02}^2 \\ & - \frac{8}{\rho} A^2 b^2 \kappa \eta_{02}^3 (1 + 2\eta_{02}^2 \kappa^2) e^{(\kappa^2(1+\frac{1}{\rho^2})\eta_{02}^2)} \\ & - \frac{8}{\rho} A^2 b^2 \kappa^3 \eta_{02}^5 e^{(\kappa^2(1+\frac{1}{\rho^2})\eta_{02}^2)} \\ & - \frac{4}{\rho} A^2 b^2 \kappa^3 \eta_{02}^5 (1 + 2\eta_{02}^2 \kappa^2) \left(1 + \frac{1}{\rho^2}\right) e^{(\kappa^2(1+\frac{1}{\rho^2})\eta_{02}^2)}. \end{aligned} \quad (58d)$$

The eigenvalues of the system (eq.(57)) determine the stability near each of the fixed points and are given by

$$\begin{aligned} \lambda_{1,2} = \pm \{ & -8\sigma^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{01}^4 + \frac{[\frac{12}{\sqrt{2}} \sigma^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{01}^4] (1+\rho^2)}{8\eta_{01} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01} \kappa e^{\vartheta_1} \rho + 4\eta_{01} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2} \\ & + \frac{[32\sqrt{\frac{\pi}{2}} \rho \sigma^2 b^2 \kappa^4 \eta_{01}^5 (1+2\eta_{01}^2 \kappa^2) e^{(\kappa^2(1+\rho^2)\eta_{01}^2)}] (1+\rho^2)}{(8\eta_{01} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01} \kappa e^{\vartheta_1} \rho + 4\eta_{01} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \\ & + \frac{[32\sqrt{\frac{\pi}{2}} \rho \sigma^2 b^2 \kappa^6 \eta_{01}^7 e^{(\kappa^2(1+\rho^2)\eta_{01}^2)}] (1+\rho^2)}{(8\eta_{01} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01} \kappa e^{\vartheta_1} \rho + 4\eta_{01} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \\ & + \frac{[32\sqrt{\frac{\pi}{2}} \rho \sigma^2 b^2 \kappa^6 \eta_{01}^7 e^{(\kappa^2(1+\rho^2)\eta_{01}^2)}] (1+\rho^2)}{(8\eta_{01} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01} \kappa e^{\vartheta_1} \rho + 4\eta_{01} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \\ & + \frac{[16\sqrt{\frac{\pi}{2}} \rho \sigma^2 b^2 \kappa^6 \eta_{01}^7 (1+2\eta_{01}^2 \kappa^2) (1+\rho^2) e^{(\kappa^2(1+\rho^2)\eta_{01}^2)}] (1+\rho^2)}{(8\eta_{01} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01} \kappa e^{\vartheta_1} \rho + 4\eta_{01} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \}^{\frac{1}{2}} \end{aligned} \quad (59a)$$

and

$$\begin{aligned}
\lambda_{3,4} = \pm \{ & -8\sigma^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{02}^4 + \frac{\left[\frac{12}{\sqrt{2}} \sigma^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{02}^4 \right] (1+\rho^2)}{8\eta_{02} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02} \kappa e^{\vartheta_1} \rho + 4\eta_{02} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2} \\
& + \frac{\left[32\sqrt{\frac{\pi}{2}} \sigma^2 b^2 \kappa^4 \eta_{02}^5 (1+2\eta_{02}^2 \kappa^2) e^{\left(\kappa^2 \left(1 + \frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02} \kappa e^{\vartheta_1} \rho + 4\eta_{02} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \\
& + \frac{\left[32\sqrt{\frac{\pi}{2}} \sigma^2 b^2 \kappa^4 \eta_{02}^5 (1+2\eta_{02}^2 \kappa^2) e^{\left(\kappa^2 \left(1 + \frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02} \kappa e^{\vartheta_1} \rho + 4\eta_{02} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \\
& + \frac{\left[32\sqrt{\frac{\pi}{2}} \sigma^2 b^2 \kappa^6 \eta_{02}^7 e^{\left(\kappa^2 \left(1 + \frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02} \kappa e^{\vartheta_1} \rho + 4\eta_{02} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \\
& + \frac{\left[16\sqrt{\frac{\pi}{2}} \sigma^2 b^2 \kappa^6 \eta_{02}^7 (1+2\eta_{02}^2 \kappa^2) \left(1 + \frac{1}{\rho^2} \right) e^{\left(\kappa^2 \left(1 + \frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02} \kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02} \kappa e^{\vartheta_1} \rho + 4\eta_{02} \kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2} \rho^2)} \}^{\frac{1}{2}}
\end{aligned} \tag{59b}$$

The eqs.(59) show that the fixed points **II** and **III** are **the centers**, regardless of the sign of the dispersion. We will then investigate both the anomalous- and normal-dispersion fibers in what follows.

A. Normal Dispersion: $\sigma = \frac{D_-}{D} < 0$

In this case, we can rewrite $\sigma = -\frac{|D_-|}{D} < 0$. We then consider the location and stability of the three fixed points of eqs.(48b), (48c), and (48d) shown in eqs.(54). Fixed point **I** has already been determined to be degenerate, and fixed points **II** and **III** lay in the left half-plane since $\eta_{0_j} < 0$. Because we consider only values of $\eta_j \geq 0$, only the fixed points at the origin are relevant. The phases-plane dynamics are shown in figures 1, 3, 5, and 7 along with the arrows in phases superposition space. We find that the shapes of the plane are the same as homoclinic orbits, which emanate and terminate in the origin.

B. Anomalous Dispersion: $\sigma = \frac{D_+}{D} > 0$

In the anomalous-dispersion regime, the phases-plane dynamics is significantly different than that of the normal-dispersion regime since fixed points **II** and **III**, given by eqs.(54b) and (54c) are located at

$$\beta_j = 0,$$

$$\text{II.} \quad \frac{\frac{1}{\sqrt{2}} A^2 b^2 \kappa^2 (1+\rho^2)}{2\kappa^4 (1+\rho^2) \frac{D_+}{D} - 2\kappa^2 \exp(\vartheta_1) [2\rho A^2 b^2 \kappa] \vartheta_1 - \kappa^2 (1+\rho^2) \exp(\vartheta_1) [2\rho A^2 b^2 \kappa]} > 0, \tag{60a}$$

and

$$\beta_j = 0,$$

$$\text{III.} \quad \frac{\frac{1}{\sqrt{2}} A^2 b^2 \kappa^2 \left(1 + \frac{1}{\rho^2} \right)}{2\kappa^4 \left(1 + \frac{1}{\rho^2} \right) \frac{D_+}{D} - 2\kappa^2 \exp(\vartheta_1) [2\rho A^2 b^2 \kappa] \vartheta_1 - \kappa^2 \left(1 + \frac{1}{\rho^2} \right) \exp(\vartheta_1) [2\rho A^2 b^2 \kappa]} > 0, \tag{60b}$$

with the four of eigenvalues

$$\lambda_{1,2} = \pm \left\{ -8 \left(\frac{D_+}{D} \right)^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{01}^4 + \frac{\left[\frac{12}{\sqrt{2}} \left(\frac{D_+}{D} \right)^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{01}^4 \right] (1+\rho^2)}{8\eta_{01}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01}\kappa e^{\vartheta_1} \rho + 4\eta_{01}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2} \right. \\ \left. + \frac{\left[32\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^4 \eta_{01}^5 (1+2\eta_{01}^2 \kappa^2) e^{\left(\kappa^2 (1+\rho^2) \eta_{01}^2 \right)} \right] (1+\rho^2)}{(8\eta_{01}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01}\kappa e^{\vartheta_1} \rho + 4\eta_{01}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right. \\ \left. + \frac{\left[32\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^6 \eta_{01}^7 e^{\left(\kappa^2 (1+\rho^2) \eta_{01}^2 \right)} \right] (1+\rho^2)}{(8\eta_{01}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01}\kappa e^{\vartheta_1} \rho + 4\eta_{01}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right. \\ \left. + \frac{\left[32\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^6 \eta_{01}^7 e^{\left(\kappa^2 (1+\rho^2) \eta_{01}^2 \right)} \right] (1+\rho^2)}{(8\eta_{01}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01}\kappa e^{\vartheta_1} \rho + 4\eta_{01}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right. \\ \left. + \frac{\left[16\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^6 \eta_{01}^7 (1+2\eta_{01}^2 \kappa^2) (1+\rho^2) e^{\left(\kappa^2 (1+\rho^2) \eta_{01}^2 \right)} \right] (1+\rho^2)}{(8\eta_{01}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{01}\kappa e^{\vartheta_1} \rho + 4\eta_{01}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right\}^{\frac{1}{2}}, \quad (61a)$$

and

$$\lambda_{3,4} = \pm \left\{ -8 \left(\frac{D_+}{D} \right)^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{02}^4 + \frac{\left[\frac{12}{\sqrt{2}} \left(\frac{D_+}{D} \right)^2 b^2 \sqrt{\frac{\pi}{2}} \kappa^3 \eta_{02}^4 \right] (1+\rho^2)}{8\eta_{02}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02}\kappa e^{\vartheta_1} \rho + 4\eta_{02}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2} \right. \\ \left. + \frac{\left[32\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^4 \eta_{02}^5 (1+2\eta_{02}^2 \kappa^2) e^{\left(\kappa^2 \left(1+\frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02}\kappa e^{\vartheta_1} \rho + 4\eta_{02}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right. \\ \left. + \frac{\left[32\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^6 \eta_{02}^7 e^{\left(\kappa^2 \left(1+\frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02}\kappa e^{\vartheta_1} \rho + 4\eta_{02}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right. \\ \left. + \frac{\left[32\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^6 \eta_{02}^7 e^{\left(\kappa^2 \left(1+\frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02}\kappa e^{\vartheta_1} \rho + 4\eta_{02}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right. \\ \left. + \frac{\left[16\sqrt{\frac{\pi}{2}} \rho \left(\frac{D_+}{D} \right)^2 b^2 \kappa^6 \eta_{02}^7 (1+2\eta_{02}^2 \kappa^2) \left(1+\frac{1}{\rho^2} \right) e^{\left(\kappa^2 \left(1+\frac{1}{\rho^2} \right) \eta_{02}^2 \right)} \right] (1+\rho^2)}{\rho (8\eta_{02}\kappa e^{\vartheta_1} \rho \vartheta_1 + 4\eta_{02}\kappa e^{\vartheta_1} \rho + 4\eta_{02}\kappa e^{\vartheta_1} \rho^3 + \sqrt{2} + \sqrt{2}\rho^2)} \right\}^{\frac{1}{2}} \quad (61b)$$

The phases-plane dynamics are shown in figures 2, 4, 6, and 8. In this case, the phases flow for $\eta_j \geq 0$ are partially determined by the three critical points **I**, **II** and **III**. It can be more obviously understood if we replace

$$C = - \left(\frac{d\phi_1(Z)}{dZ} + \frac{d\phi_2(Z)}{dZ} \right),$$

in the eqs.(50), We then give rise to the separatrix

$$\beta_1 = \pm \frac{\eta_1}{\sqrt{2}} \left\{ \frac{[2+2\rho]\overline{D}\eta_1}{D_+\kappa\sqrt{\pi}} + \frac{4\sqrt{2}\rho\overline{D}\eta_1^2 \exp[\kappa^2(1+\rho^2)\eta_1^2]}{D_+\sqrt{\pi}} - (\rho^2 + 1) \eta_1^2 \right\}^{\frac{1}{2}}, \quad (62a)$$

$$\beta_1 = \pm \frac{\eta_2}{\rho\sqrt{2}} \left\{ \frac{[\frac{2}{\rho}+2]\overline{D}\eta_2}{D_+\kappa\sqrt{\pi}} + \frac{\overline{D}4\sqrt{2}\eta_2^2 \exp[\kappa^2(\frac{1}{\rho^2}+1)\eta_2^2]}{D_+\sqrt{\pi}} - \left(1 + \frac{1}{\rho^2} \right) \eta_2^2 \right\}^{\frac{1}{2}}, \quad (62b)$$

$$\beta_2 = \pm \left(\frac{\rho}{\sqrt{2}} \right) \eta_1 \left\{ \frac{[2+2\rho]\overline{D}\eta_1}{D_+\kappa\sqrt{\pi}} + \frac{4\sqrt{2}\rho\overline{D}\eta_1^2 \exp[\kappa^2(1+\rho^2)\eta_1^2]}{D_+\sqrt{\pi}} - (\rho^2 + 1)\eta_1^2 \right\}^{\frac{1}{2}}, \quad (62c)$$

and

$$\beta_2 = \pm \frac{\eta_2}{\sqrt{2}} \left\{ \frac{[\frac{2}{\rho}+2]\overline{D}\eta_2}{D_+\kappa\sqrt{\pi}} + \frac{\overline{D}4\sqrt{2}\eta_2^2}{\rho} \exp\left[\kappa^2\left(\frac{1}{\rho^2}+1\right)\eta_2^2\right] - \left(1 + \frac{1}{\rho^2}\right)\eta_2^2 \right\}^{\frac{1}{2}}, \quad (62d)$$

which have a cusp at the origin ($\beta_j = \eta_j = 0$). The solutions outside these separatix eventually flow into the origin (homoclinic orbits), while those inside the separatix are periodic.

From figures 2, 4, 6, and 8 in which the parameter ρ has been set to be, for instance, $\rho = 0.03$, we then find exactly that the fixed points of $(\eta_1, \beta_1 = 0)$, $(\eta_2, \beta_1 = 0)$, $(\eta_1, \beta_2 = 0)$ and $(\eta_2, \beta_2 = 0)$ are at the center of their graphics as follows

$$(\eta_1, (\beta_1 = 0)) = (\eta_1, (\beta_2 = 0)) = (0.75, 0), \quad (63a)$$

and

$$(\eta_2, (\beta_1 = 0)) = (\eta_2, (\beta_2 = 0)) = (0.0225, 0). \quad (63b)$$

The points and the phases-plane dynamics of the **ODEs** eqs.(48b), (48c) and (48d) improve that our suggestion in eq.(31c), $\rho(Z) = \frac{\beta_2}{\beta_1} = \frac{\eta_2}{\eta_1} = \text{const.}$, is actually proved. This means the coupled breathers dynamics depend on the certain relationship of the chirps and amplitudes. The choice of an arbitrary $\rho(Z)$ makes the certain fixed points of (η_j, β_j) obeyed by eqs.(54). These fixed points are found in $\left(C + \frac{d\phi_1(Z)}{dZ} + \frac{d\phi_2(Z)}{dZ}\right) \approx 0.605$. In the physical meaning, the interactions between both chirps depend on a linear constant $\rho(Z)$. However, the magnitude of the Gaussian pulses amplitudes is not only directly influenced by the chirp itself but also by the interaction of both chirps.

On the other hand, we, for instance, can set the values of $\rho(Z)$ and then find the the amplitudes by using a numerical method from their phases-plane dynamics graphics.

A geometrical representation of the solution and its associated dynamics behavior of eqs.(48b), (48c) and (48d) appeared in figures 1-8 show the superposition of the flow of phases (ϕ_1 and ϕ_2). According to the figures, the construction of a periodic dispersion-managed coupled breathers solution from the dispersion map depends on the critical values of A_j . In this periodic solutions, the pulses first evolve according to the first segments of the dispersion map for which $\sigma_j < 0$ (figures 1, 3, 5, and 7). The dynamics then reverse flow according to the dispersion switches sign as shown in figures 2, 4, 6, and 8, and then finally are again subjects to the dynamics. This case is in a reflection symmetry of $\beta_j = 0$ derived from the choice of dispersion map and the resulting invariance

of eqs.(48b), (48c) and (48d) under the following reversibility transformation $(\eta_j, \beta_j, Z) \rightarrow (\eta_j, -\beta_j, -Z)$. The qualitative descriptions are powerful enough when the exact values of the fixed points (η_j, β_j) (eqs.(63)) are directly found in the phases-plane dynamics graphics, and have been possibly proved in this paper.

For the Single **NLS** investigated before by Kutz, et al., all explanations about the Hamiltonian dynamics of the dispersion-managed breathers can be derived from the reduced **ODEs** of the coupled **NLS** in **section 6**. For instance, the two fixed points and the eigenvalues which are easily found by reducing eqs.(48) and (59) are as follows

$$\text{I. } \beta_1 = 0, \quad \eta_1 = 0, \quad (64a)$$

$$\text{II. } \beta_1 = 0, \quad \eta_2 = 0, \quad \eta_1 = \frac{a_{21}a_{71}}{a_{71}a_{11}} = \eta_{10}, \quad (64b)$$

where $\left(b^2 = \frac{2\kappa}{\sqrt{\frac{\pi}{2}}}\text{and } \rho = 0\right)$,

$$\eta_1 = \frac{a_{21}a_{71}}{a_{71}a_{11}} = \frac{A_1^2}{\sqrt{\pi}\kappa\sigma_1(Z)} = \eta_{01}, \quad (65)$$

and

$$\lambda_{1,2} = \pm i2\sigma\kappa^2\eta_{01}^2 \quad (66)$$

VII. Conclusion

In conclusion, we have presented the recent results related to Hamiltonian Dynamics of Dispersion-Managed Coupled Breathers in Optical Transmission System. From mathematical viewpoints, the use of a variational method and its corresponding Hamiltonian formulation of the coupled **NLS**, the pulse dynamics in a dispersion-managed optical system can be reduced to four nonlinear **ODEs**. By means of the conserved quantity of the Hamiltonian system, an exact solution of both β_j and η_j can also be found in terms of a modified quadrature representations. The results of the coupled **NLS** with the perturbation term can also be reduced to both that system which has no the perturbation term and the single **NLS** case. From the physical viewpoint, we conclude that a noise sourced by the amplifier which has a small frequency causes a small change in the coupled breathers dynamics of the Gaussian pulses. We then get some phases-plane dynamics of the reduced system in several fixed points. We have also shown that the linearization and use of the Hamiltonian integral of the four **ODEs** describe the dynamics of the phases-plane and further suggests the possibility of constructing a periodic solution that depends on the initial amplitude-enhancement factor of the pulse power. Finally, we then find the

proof from the phases-plane dynamics that the coupled breathers dynamics depend on the certain relationship (eq.(31c)) and the parameter A_j , of the chirps and amplitudes.

Thus, based on our results, there are some important open problems related to the another coupled **NLS** with a complicated loss and gain terms. However, the dispersion-managed vector breathers dynamics derived from the Gaussian pulses as a solution of the nonintegrable vector **NLS** :

$$i \frac{\partial U_j}{\partial Z} + \frac{\sigma_j(Z)}{2} \frac{\partial^2 U_j}{\partial T^2} + \left(\sum_{j=1}^N |U_j|^2 \right) U_j = 0,$$

where $\sigma_1(Z) = \dots = \sigma_N(Z) = \sigma(Z)$

is in progress. As a result, the advantages for the application of the Gaussian pulses in optical communication system instead of the initial small chirps and pulses energy are still open.

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